

TH-FORKING, ALGEBRAIC INDEPENDENCE AND EXAMPLES OF ROSY THEORIES.

ALF ONSHUUS

1. INTRODUCTION

In [Ons] we developed the notions of \mathfrak{p} -independence and \mathfrak{p} -ranks which define a geometric independence relation in a class of theories which we called “rosy”. We proved that rosy theories include simple and o-minimal theories and that for any theory for which the stable forking conjecture was true, \mathfrak{p} -forking coincides with forking independence.

In this article, we continue to study properties of \mathfrak{p} -forking and examples of rosy theories. In the first section we recall the definitions and basic results proved in [Ons]. In section 2 we study alternative ways to characterize rosy theories and we prove the coordinatization theorem, which will prove very useful when studying examples of superrosy theories. In section 3 we prove that given any theory T with weak elimination of imaginaries and in which algebraic independence defines a geometric independence relation, T is rosy and \mathfrak{p} -forking agrees with algebraic independence. Finally, in section 4 we study two examples of rosy theories. We prove that pseudo real closed fields (PRC-fields) are rosy and that whenever one has a large differential field whose restriction to the language of rings is a model complete rosy field, the model companion (as defined by Tressl in [Tre]) is also rosy. This is an extension of the result we had in [Ons02] for closed ordered differential fields.

Throughout this article we assume familiarity with the terminology, definitions and basic results in stability and simplicity theory. We work with first order theories which will generally be denoted by T . As is usual in stability theory, unless otherwise specified, we work inside a monster model \mathcal{C}^{eq} of T . By convention lower case letters a, b, c, d will in general represent tuples (of imaginaries) unless otherwise specified, and upper case letters will represent sets. Greek letters such as $\delta, \sigma, \psi, \phi$ will be used for formulas. Given a set A and a tuple a , the type of a over A will be denoted by $tp(a/A)$ and we will abbreviate “ $tp(a/Ab)$ does not fork over A ” as $a \perp_A b$.

1.1. Preliminaries and notation. We recall the definitions of \mathfrak{p} -forking and rosy theories and the main results proved in [Ons]. The main definitions we work with are the following:

Definition 1.1. A formula $\delta(x, a)$ strongly divides over A if $tp(a/A)$ is non-algebraic and $\{\delta(x, a')\}_{a' \models tp(a/A)}$ is k -inconsistent for some $k \in \mathbb{N}$.

We say that $\delta(x, a)$ \mathfrak{p} -divides over A if we can find some tuple c such that $\delta(x, a)$ strongly divides over Ac .

A formula \mathfrak{p} -forks over A if it implies a (finite) disjunction of formulae which \mathfrak{p} -divide over A .

As is done with the standard forking, we say that the type $p(x)$ \mathfrak{p} -divides over A if there is a formula in $p(x)$ which \mathfrak{p} -divides over A ; \mathfrak{p} -forking is similarly denoted. We say that a is \mathfrak{p} -independent of b over A , denoted $a \perp_A^{\mathfrak{p}} b$, if $\text{tp}(a/Ab)$ does not \mathfrak{p} -fork over A .

As we mentioned before, \mathfrak{p} -forking defines an independence relation in a large class of theories called rosy theories which includes simple and o-minimal structures. Before we are able to give the definition of rosy theories we must first define the class of ranks that is associated with \mathfrak{p} -forking.

Definition 1.2. Given a formula ϕ , a set Δ of formulas in the variables $x; y$, a set of formulae Π in the variables $y; z$ (with z possibly of infinite length) and a number k , we define $p(\phi, \Delta, \Pi, k)$ inductively as follows:

- (1) $p(\phi, \Delta, \Pi, k) \geq 0$ if ϕ is consistent.
- (2) For λ limit ordinal, $p(\phi, \Delta, \Pi, k) \geq \lambda$ if and only if $p(\phi, \Delta, \Pi, k) \geq \alpha$ for all $\alpha < \lambda$
- (3) $p(\phi, \Delta, \Pi, k) \geq \alpha + 1$ if and only if there is a $\delta \in \Delta$, some $\pi(y; z) \in \Pi$ and parameters c such that
 - (a) $p(\phi \wedge \delta(x, a), \Delta, \Pi, k) \geq \alpha$ for infinitely many $a \models \pi(y; c)$
 - (b) $\{\delta(x, a)\}_{a \models \pi(y; c)}$ is k -inconsistent

A theory is called “rosy” if all of its \mathfrak{p} -ranks are defined. It is clear from the definitions that all simple theories are rosy (for any \mathfrak{p} -rank we can easily find a D-rank such that the value is bigger for all formulas). We also have the following theorem.

Theorem 1.1.1. *In a rosy theory \mathfrak{p} -forking has all the properties of a geometric independence relation.*

We should point out that the independence theorem does not always work for \mathfrak{p} -forking (as evidenced by the fact that it does define an independence relation in o-minimal theories) and we therefore cannot conclude that \mathfrak{p} -forking is the same as forking when restricted to simple theories. We do know that in all stable and supersimple theories (and in fact in any simple theory for which \mathfrak{p} -forking satisfies the stable forking conjecture) forking is the same as \mathfrak{p} -forking¹.

2. CHARACTERIZATION OF ROSY THEORIES AND COORDINATIZATION THEORY

In this section we study alternative ways of characterizing \mathfrak{p} -forking. We prove that symmetry and local character of \mathfrak{p} -forking are both characteristics

¹Clifton Ealy has extended this result to all simple theories that satisfy elimination of hyperimaginaries

that imply rosiness. We also prove the coordinatization theorem (proved by Hart, Kim and Pillay - [HKP00]- for simple theories).

Remark 2.0.2. *Given any type p over some set $B \supset A$, if p is finitely satisfiable over A then it does not p -fork over A .*

Proof. We know that if a type is finitely satisfiable it does not fork over A . *A fortiori*, it cannot p -fork. \square

For the proofs in this section we use repeatedly the following result. The proof is very similar to its simple theoretic analogue ([Wag00] 2.3.7).

Lemma 2.0.3. *Let $p(x = x, \phi, \pi, k) \geq n$ for all $n < \omega$. Then for every linearly ordered index set I , there are*

- (1) *An indiscernible sequence $(b_i a_i : i \in I)$ such that $\models \phi(b_i, a_j)$ for any $j \leq i$ and $\phi(x, a_i)$ p -forks over $\{a_j \mid j < i\}$.*
- (2) *A tuple b and a b -indiscernible infinite sequence $\langle a_i \rangle$ such that $\models \phi(b, a_i)$ and $\phi(x, a_i)$ p -forks over $\{a_j \mid j < i\}$.*

Proof. 1. By definition of p -rank, given $p(x)$ a partial type over A , if $p(p(x), \phi, \pi, k) \geq n+1$ we can find tuples a, b, c such that $\models \phi(b, a) \wedge \pi(a, c)$, $tp(a/Ac)$ is non algebraic, $p(p(x) \cup \{\phi(x, a)\}, \phi, \pi, k) \geq n$ and $\{\phi(x, a')\}_{a' \models \pi(y, c)}$ is k -inconsistent. Thus, for every n , we can find by induction tuples $\langle a_i^n, b_i^n, c_i^n \rangle_{i \leq n}$ such that $b_i^n \models \phi(x, a_j^n)$ for any $j \leq i$, $\{\phi(x, a')\}_{a' \models \pi(y, c_i^n)}$ is k -inconsistent, $tp(a_i^n / Ac_i^n a_0^n, a_1^n, \dots, a_{i-1}^n)$ is non-algebraic and

$$p \left(\bigcup_{j \leq i} \{\phi(x, a_j^n)\}, \phi, \pi, k \right) \geq n - i.$$

By compactness we can build a similar sequence of any given length and using Ramsey's theorem and compactness we can find one such sequence such that $\langle a_i b_i \rangle$ is indiscernible.

2. Is a particular case of the sequence we get above with $b = b_\omega$. \square

Remark 2.0.4. *We are actually proving something slightly stronger. If in the previous lemma we assume that for some type $p(x)$, $p(p(x), \phi, \pi, k) > n$ for all $n < \omega$, then we can actually get all the b 's and b_i 's in the conclusions satisfying $p(x)$.*

2.1. Characterizations by Local Character and Symmetry.

Theorem 2.1.1. *A theory is rosy if and only if p -forking satisfies local character.*

Proof. To prove the left to right implication, assume that all the p -ranks are finite. Fixing a type p , for any formulas ϕ, π and any integer k , there must be some finite tuple $a_{\phi\pi k}$ such that $p(p, \phi, \pi, k) = p(p \upharpoonright a_{\phi\pi k}, \phi, \pi, k)$. Then, if we let A be the union of all such tuples, $|A| \leq |T|$ and p does not p -fork over A .

For the other direction, assume that $D(x = x, \phi, \pi, k) \not\prec \omega$. By 2.0.3 there is some b and some b -indiscernible sequence $\langle a_i \rangle_{i \in |T|^+}$ such that for any $i \in |T|^+$, $\phi(x, a_i)$ \mathfrak{p} -divides over $\{a_j \mid j < i\}$ and $b \models \phi(x, a_i)$. Let $A = \{a_i \mid i \in |T|^+\}$. Now, given any $A_0 \subset A$ of cardinality $|T|$, let λ be some ordinal in $|T|^+$ such that $A_0 \subset \{a_j \mid j < \lambda\}$. Then $\phi(b, a_\lambda) \in tp(b/A)$ and it \mathfrak{p} -forks over A_0 , contradicting local character for $tp(b/A)$. \square

Theorem 2.1.2. *A theory is rosy if and only if \mathfrak{p} -forking satisfies symmetry.*

Proof. The left to right implication follows from \mathfrak{p} -forking being a geometric independence relation in rosy theories. For the other implication, let us assume again that $D(x = x, \phi, \pi, k) \not\prec \omega$ and by 2.0.3 there is some b , some b -indiscernible sequence $\langle a_i \rangle_{i \leq \omega}$ and some $\phi(x, y)$ such that for all i we have that $b \models \phi(x, a_i)$ and $\phi(x, a_i)$ \mathfrak{p} -divides (and therefore it \mathfrak{p} -forks) over $\{a_j \mid j < i\}$. By indiscernibility, $tp(a_\omega/ba_1, \dots, a_n, \dots)$ is finitely satisfiable over a_1, \dots, a_n, \dots and thus it does not \mathfrak{p} -fork over a_1, \dots, a_n, \dots . On the other hand $b \models \phi(x, a_\omega)$ so in any non-rosy theory, \mathfrak{p} -forking is not symmetric. \square

2.2. Coordinatization. All the definitions and results in this subsection analogues of simple theoretic results proved in [HKP00]; the proofs are very similar.

Definition 2.1. *We say that a theory T is finitely coordinatized by a set of types \mathcal{P} if \mathcal{P} is closed under automorphisms and for any type $p(x)$ over a tuple a there is some $n \in \omega$ and a sequence a_0, \dots, a_n with $a_0 = \emptyset$, $a_n = a$ and $tp(a_i/a_{i-1}) \in \mathcal{P}$.*

Definition 2.2. *Given any type $p \in S(T)$, we say that p is rosy if all of its extensions satisfy local character for \mathfrak{p} -forking.*

Lemma 2.2.1. *Let $p(x)$ be any type. Then $\mathfrak{p}(p(x), \delta, \pi, k) < \omega$ for all formulas δ , π and all $k \in \mathbb{N}$ if and only if $p(x)$ is rosy.*

Proof. Using 2.0.4 the result follows from the proof of 2.1.1. \square

Theorem 2.2.2. *If T is coordinatized by rosy types, then it is rosy.*

Proof. Let a be any tuple, and A be a set; we can assume a itself is a coordinatizing sequence (after maybe adding entries to a) of size, say, n . We will prove that for any ϕ, π, k , $\mathfrak{p}(tp(a/A), \phi, \pi, k)$ is finite by induction on n . If $n = 1$ the result follows from 2.2.1. So let us assume $a = bc$ where b is a (coordinatizing) sequence of size $n - 1$ and c is a singleton. By induction hypothesis, there is some $A_0 \subset A$ of size less than $|T|$ such that $tp(b/A)$ does not \mathfrak{p} -fork over A_0 . By local character (maybe after increasing A_0 but keeping its size smaller than $|T|$) we can assume $tp(c/Ab)$ does not \mathfrak{p} -fork over A_0b . By partial left transitivity ([Ons] lemma 2.1.5) we get that $tp(bc/A)$ does not \mathfrak{p} -fork over A_0 . \square

Definition 2.3. *A theory T is \mathfrak{p} -minimal if for any model $M \models T$, any $a \in M^1$ and any $A \subset M$, $tp(a/A)$ \mathfrak{p} -forks over \emptyset if and only if a is algebraic over A .*

Corollary 2.2.3. *Given a complete theory T , let $M \models T$ be a model of T . Then, if for all elements $a \in M^1$ and all sets $A \subset M$ $U^b(a/A) \leq 1$, T is superrosy. In particular, any b -minimal theory is superrosy.*

Proof. Clearly the set of all types of single elements coordinatizes the theory. Being b -minimal is equivalent to have the U^b -rank of any such type (and therefore all the b -ranks) be at most 1 and by 2.2.1 any type in $S_1(T)$ is rosy. By theorem 2.2.2 T is rosy and by Lascar's inequalities for rosy theories it is in fact superrosy.

Finally, let T be b -minimal, $p \in S_1(T)$ be a type over A , q be a type over $B \supset A$ extending p and let a be an element satisfying q . If q is not algebraic, then q does not b -fork over the empty set and it is rosy. If q is algebraic, then there is some finite $C \subset B$ such that $tp(a/C)$ is algebraic and q does not b -fork over C . \square

3. b -FORKING AND ALGEBRAIC INDEPENDENCE IN NON-SATURATED MODELS

Besides simple theories, most examples of theories with an independence relation are those for which algebraic independence satisfies the Steinitz exchange property. In this section we prove that if algebraic independence is a geometric independence relation and one has elimination of imaginaries the theory is rosy and algebraic independence corresponds to b -forking.

This results were proved in [Ons02] when we were studying what (weakened) version of amalgamation of independent types one could prove for b -forking in a general rosy theory. It is known (see for example [Kim96]) that the only independence relation satisfying symmetry, local character, transitivity and having amalgamation for independent types (the independence theorem) was the non forking relation inside a simple theory. This means that the only rosy theories in which b -forking can have the independence theorem are simple theories in which b -forking agrees with forking.

However, in our main non-simple examples (o-minimal theories) the following amalgamation theorem is true:

Consistent Amalgamation. Let $p(x)$ be a complete type over some set A . Let $p(x, a)$ and $q(x, b)$ be two non- b -forking extensions of $p(x)$ such that $a \perp_A^b b$. Then, either $p(x, a) \cup p(x, b)$ is inconsistent, or it is a non- b -forking extension of $p(x)$.

All the examples of rosy theories mentioned here have consistent amalgamation. The results in this section were used to find an example of a rosy theory that does not have consistent amalgamation. The details of the construction can be found in [Ons02].

3.1. b -forking in non-saturated models. Up until now we have followed the simple theoretic approach of working inside a large saturated model $C \models T$. In simple theories this is necessary to understand the behavior of forking-independence in T , mainly because the definition of forking requires

indiscernible sequences and the existence of such sequences varies inside different models of T . We prove that, even though we still need some saturation to fully understand the behavior of \mathfrak{p} -forking, the amount of saturation we need is far from what we need to study simple theories.

Definition 3.1. *Let T be a theory, M be a model of T and \mathcal{C} be a monster model of T containing M .*

Given a set A in M , the algebraic closure of A in M , which we denote as $\text{acl}^M(A)$, is the set of elements b in M such that $\text{tp}(b/A)$ has infinitely many realizations in M .

A model $M \models T$ is weakly ω -saturated if for any $c \in M$ any formula of the form

$$\phi(x_1, x_2, \dots, x_n; c)$$

if there are a_1, a_2, \dots, a_n in \mathcal{C} such that $\mathcal{C} \models \phi(a_1, \dots, a_n; c)$ and for any i $a_i \notin \text{acl}(a_{i+1}, \dots, a_n, c)$ then there are b_1, b_2, \dots, b_n in M such that $M \models \phi(b_1, \dots, b_n; c)$ and for any i $b_i \notin \text{acl}^M(b_{i+1}, \dots, b_n, c)$.

Theorem 3.1.1. *Let M be a model of T that is weakly ω -saturated. Then \mathfrak{p} -forking dependence (between tuples in M) is always witnessed by elements in M .*

This is, if $a, b \in M$, $A \subset M$ and $a \not\mathcal{J}_A^{\mathfrak{p}} b$ then there are formulas $\psi_i(x, b_i)$, $b_i \in M$ and tuples $c_i \in M$ such that

- $\text{tp}(a/Ab) \models \bigvee_{i=1}^m \psi_i(x, b_i)$,
- $\{\psi_i(x, b'_i)\}_{b'_i \models \text{tp}(b_i/Ac_i)}$ is k -inconsistent and
- $\text{tp}(b_i/Ac_i)$ has infinitely many realizations in M .

Proof. k -inconsistency can be witnessed by a formula so all three of the statements follow from the definition of \mathfrak{p} -forking and weak ω -saturation. \square

Corollary 3.1.2. *A theory T is \mathfrak{p} -minimal if and only if for any weakly ω -stable model M , any tuples $a, b \in M^1$ and any finite $A \subset M$, $a \not\mathcal{J}_A^{\mathfrak{p}} b$ if and only if $\text{tp}(a/Ab)$ is algebraic in M and $\text{tp}(a/A)$ has infinitely many realizations (in M).*

3.2. \mathfrak{p} -Forking and algebraic independence.

Definition 3.2. *Following Pillay, we say that a theory T has geometric elimination of imaginaries if for any model $M \models T$ and any imaginary element $e \in \text{dcl}^{\text{eq}}(M)$, there is some finite $x \subset M$ such that $\text{acl}^{\text{eq}}(e) = \text{acl}^{\text{eq}}(x)$. We say that a model M weakly eliminates imaginaries if $\text{Th}(M)$ does.*

Theorem 3.2.1. *Let M be a weakly ω -stable model which satisfies the Steinitz exchange property for the algebraic closure and which has weak elimination of imaginaries. Then $\text{Th}(M)$ is \mathfrak{p} -minimal and has global $U^{\mathfrak{p}}$ -rank 1. Moreover, $U^{\mathfrak{p}}(A) = \dim^{\text{alg}}(A)$.*

Proof. By 2.2.3 and 3.1.2 it is enough to show that for any $a \in M^1$ and any $A \subset M$, $U^{\mathfrak{p}}(a/A) \leq 1$.

We proceed by contradiction. Suppose we have some $b \in M^1$ such that $U^b(b) \geq 2$. By definition there are finite $A, C \subset M$ such that $p(x, A) := tp(b/A)$ is non-algebraic and there is some formula $\phi(x, A) \in p(x, A)$ such that

$$\{\phi(x, A')\}_{A' \models tp(A/C)}$$

is k -inconsistent and $tp(A/C)$ is non-algebraic. Using weak elimination of imaginaries, we may replace A and C by tuples of elements in M^1 .

Claim 3.2.2. *We may assume without loss of generality that b is not algebraic over $A \cup C$.*

Proof. Since $\phi(C, A)$ is non algebraic it is unbounded so there is some $b' \in C$ satisfying $\phi(x, A)$ and such that b' is not algebraic over $A \cup C$. By weak ω -saturation, we can find such a b in M . \square

$\{\phi(x, A')\}_{A' \models tp(A/C)}$ is k -inconsistent so $tp(A/Cb)$ is algebraic. Since $tp(A/C)$ is not algebraic by definition and A and C are tuples, we can apply Steinitz exchange property repeatedly and eventually we get that $tp(b/AC)$ is algebraic, a contradiction. \square

4. TWO EXAMPLES OF ROSY THEORIES

4.1. Pseudo Real Closed Fields. In [GvdDM81], Cherlin, van den Dries and Macintyre proved that the theory of a bounded pseudo-algebraically closed field² is decidable and therefore behaves nicely in a model theoretic sense. In a series of papers (see [Hru91], [Cha99] and [CP98]), Hrushovski, Chatzidakis and Pillay proved that any such theory (the theory of a bounded pseudo-algebraically closed field) is simple; even more, they proved that a pseudo-algebraically closed field is simple if and only if it is bounded.

In [Pre81], Prestel defined the theory of pseudo real closed fields (PRC-fields) which includes ordered (or orderable) fields that are “close” to being real closed fields (in the same sense in which PAC fields are “close” to being algebraically closed). Prestel’s definition states that a field F is a PRC-field if any absolutely irreducible variety that has a K -rational point in every real closed field K containing F , has an F -rational point. This definition is a strict generalization of PAC-fields (there are no real closed fields containing a PAC-field so the condition of having a rational point in every real closure becomes vacuous) and works nicely when trying to describe the possible absolute Galois groups of a PRC-field.

We work with a definition closer to the one used in [Hru91] for PAC-fields. We work inside a two sorted structure; let (D, F) be a pair of ordered fields such that D is the real closure of F and such that if V is an open cell in D^n such that the (topological) closure of V is defined as the “real” zeros of an

²A field F is pseudo-algebraically closed (PAC) if and only if any absolutely irreducible variety defined by polynomials with coefficients in F has an F -rational point. A PAC field is bounded if the absolute Galois group is small: this is, if for any $n \in \mathbb{N}$ the number of elements in $Gal(acl(F) : F)$ of degree n is finite.

absolutely irreducible variety definable over F , then V has F -rational points. Let $T = Th((D, F))$.

The first thing we show is that this definition is not far from the one stated in [Pre81].

Claim 4.1.1. *For any PRC-field (in the sense of Prestel) F , if D is the real closure of F , then $(D, F) \models T$.*

Proof. In [Sch00] Theorem 2.1 Schmid proved that if F is a PRC-field and V is an absolutely irreducible affine variety definable over F then the set of F -rational points in V is a dense subset (in the order topology) of the set of D -rational points in V . The claim follows. \square

In the rest of the section we prove that if F is a bounded PRC field, then T is rosy. In fact we prove the following theorem.

Theorem 4.1.2. *Let F be a bounded, PRC field. Let T^* be $Th((F, a)_{a \in N})$ where N is an elementary submodel of F . Then T^* has elimination of imaginaries and algebraic closure satisfies Steinitz exchange property.*

Both the proofs of elimination of imaginaries and Steinitz exchange property rely on the proofs given by Hrushovski for perfect PAC fields in [Hru91].

Lemma 4.1.3. *Let F be a PRC field, let D be its real closure and let $i = \sqrt{-1}$. Then $F(i)$ is a perfect PAC-field and $D(i)$ is its algebraic closure.*

Proof. Let V be an absolutely irreducible variety definable over $F(i)$. We want to show that V has $F(i)$ -rational points.

Being irreducible is a property that can be determined by looking at the restrictions of V inside an open affine cover, so we can assume that V is an affine variety, $V = Spec(F(i)[\bar{x}]/I)$ where I is an ideal generated by polynomials $\{p_1, \dots, p_m\}$.

For any $a \in F(i)$ let a^c be the complex conjugate of a and let a^r, a^{im} be elements in F such that $a = a^r + ia^{im}$. We define p_i^c , V^c and V^R in the following way. Let $\bar{a} := \langle a_i \rangle_{i \leq n}$ be a tuple in $F(i)$. Let V^c be the variety such that for all $a, \bar{a} \in V$ if and only if $\bar{a}^c := \langle a_i^c \rangle_{i \leq n}$ is a tuple in V^c ; let p_i^c be a polynomial such that for all x , $p_i(x) = 0$ if and only if $p_i^c(x^c) = 0$ so that

$$V^c = Spec(F(i)[\bar{x}]/\langle p_1^c, \dots, p_m^c \rangle).$$

Finally, let V^R be the smallest variety such that $\bar{a} \in V$ if and only if

$$\overline{a^R} := \langle a_i^r, a_i^{im} \rangle_{i \leq n} \in V^R.$$

In particular, if a tuple $\langle a_j, b_j \rangle$ is in V^R then for any $q(x) \in I$, $q(\langle a_j \rangle + i \langle b_j \rangle) = 0$ and $q^c(\langle a_j \rangle - i \langle b_j \rangle) = 0$.

Claim 4.1.4. *The map*

$$\sigma : V \times V^c \rightarrow F(i)^{2n}$$

defined by sending

$$(\langle a_j \rangle, \langle a_j^c \rangle) \text{ to } \left(\frac{\langle a_j \rangle + \langle a_j^c \rangle}{2}, \frac{\langle a_j \rangle - \langle a_j^c \rangle}{2i} \right) = (\langle a_j^r, a_j^{im} \rangle_{j \leq n})$$

is an isomorphism from $V \times V^c$ onto V^R .

Proof. The composition of the natural map $V \rightarrow V \times V^c$ with σ sends a tuple \bar{a} in V to \bar{a}^R . By definition the image of σ contains V^R .

Now, given any tuple $\langle a_j, b_j \rangle_{i \leq n}$ in V^R , consider the map τ that sends $\langle a_j, b_j \rangle$ to the pair $(\langle a_j + ib_j \rangle, \langle a_j - ib_j \rangle)$. For any $q(x) \in I$, $q(a_j + ib_j) = 0$ and $q(a_j - ib_j) = 0$; by definition, $(\langle a_j + ib_j \rangle, \langle a_j - ib_j \rangle) \in V \times V^c$ and taking the composition of the two maps we find that τ is the inverse of σ . \square

Corollary 4.1.5. *If V is irreducible, so is V^R .*

Proof. V^c is isomorphic to V and therefore irreducible. The product of two irreducible varieties is irreducible. \square

To finish the proof of the lemma, we need to show that V^R is definable over F .

Claim 4.1.6. *For any a, b , $\frac{b^c + a^c}{2} = \left(\frac{a+b}{2}\right)^c$ and $\frac{b^c - a^c}{2i} = \left(\frac{a-b}{2i}\right)^c$.*

Proof. The first equality in the claim is clear. For the second part, a simple calculation shows that for any $d \in D(i)$, $(d/i)^c = (-d^c)/i$; taking $d = a - b$ proves the claim. \square

Corollary 4.1.7. *Given any \bar{a} and \bar{b} , if (\bar{a}, \bar{b}) is in V^R then the conjugate $(\overline{a^c}, \overline{b^c})$ is in V^R and therefore V^R is definable over F .*

Proof. Since addition and multiplication by constants can be done in each coordinate of a tuple, we can apply claim 4.1.6 to \bar{a} and \bar{b} . By 4.1.4 there are (\bar{c}, \bar{d}) in $V \times V^c$ such that

$$(\bar{a}, \bar{b}) = \left(\frac{\bar{c} + \bar{d}}{2}, \frac{\bar{c} - \bar{d}}{2i} \right).$$

By definition, $(\overline{d^c}, \overline{c^c})$ is also in $V \times V^c$ and using 4.1.4 once again,

$$\left(\frac{\overline{d^c} + \overline{c^c}}{2}, \frac{\overline{d^c} - \overline{c^c}}{2i} \right) = (\overline{a^c}, \overline{b^c})$$

is in V^R . By hypothesis $V^R \cong V \times V^c$ is definable over $F(i)$ so V^R is in fact definable over F . \square

To finish the proof of lemma 4.1.3 note that $D(i)$ is algebraically closed so V has $D(i)$ -rational points. Taking the image of such points in V^R (the real and imaginary components of the $D(i)$ -rational points contained in V) we get that $V^R(D)$ is non-empty. By definition of PRC-fields there is some F -rational point $\langle a_j, b_j \rangle$ in V^R and $\langle a_j + ib_j \rangle$ is an $F(i)$ -rational point in V .

Since V was any irreducible $F(i)$ -definable variety, every absolutely irreducible variety definable over $F(i)$ has $F(i)$ -rational points. \square

To prove theorem 4.1.2 we follow the proofs of weak elimination of imaginaries for PAC-fields (corollary 3.2 in [Hru91]) and of corollary 1.9 in [Hru91].

Let \mathcal{L} be the language of ordered fields, let \mathcal{L}_- be the language of fields and let \mathcal{L}_+ be the language of ordered fields with a predicate F representing a pseudo real closed subfield of D . T can then be interpreted as a theory in the language \mathcal{L}_+ . Notice that for any model $M := (D, F) \models T$, the theory $Th^{\mathcal{L}}(M)$ is the theory of real closed fields. From now on we only work with bounded pseudo real closed fields and T is understood to be the theory of a bounded PRC-field.

For any element a and any set B in D , let $tp(a/B)$ and $tp_+(a/B)$ be the type of a over B in the languages of \mathcal{L} and \mathcal{L}_+ respectively.

We define the *field-definable closure* in the following way: given some set A , the field definable closure of A ($dcl^{field}(A)$) is the set of elements that are definable using quantifier free formulas in the language $\mathcal{L}_-(A)$. Note that any field-definable element can be seen as the only element of an irreducible variety, so F is field-definably closed.

Let $M := (D, F)$ be a model of the theory T (with F a bounded PRC-field) and let $M(i) := (D(i), F(i))$. A substructure M_0 of M is said to be *full* if it is algebraically closed and $acl_{\mathcal{L}_-}(F^M) \subseteq dcl^{field}(F^M \cup acl_{\mathcal{L}_-}(F^{M_0}))$.

Remark 4.1.8. *If M_1 is a model of T and M_0 is a full submodel of M_1 , then $M_0(i)$ is a full submodel of $M_1(i)$.*

Proof. We need to show that $M_0(i)$ is a submodel of $M_1(i)$, that $D_j(i)$ is algebraically closed for $j = 0, 1$ and the “fullness” condition. Both D_0 and D_1 are real closed fields so both $D_0(i)$ and $D_1(i)$ are algebraically closed.

The other conditions follow from the fact that all the field operations inside $M_j(i)$ (and therefore the theory) are interpretable inside M_j using only quantifier free formulas in \mathcal{L}_- . \square

Remark 4.1.9. *If M_1 is a model of T and M_0 is a submodel of M_1 , then M_0 is a full submodel of M_1 .*

Proof. Let $M_0 = (D_0, F^{M_0})$ be a submodel of M_1 . By 4.1.3, $M_0(i)$ is a submodel of $M_1(i)$ and by [Hru91] it is a full submodel so

$$M_1(i) \subseteq dcl^{field}(F^{M_1(i)} \cup acl(F^{M_0(i)}))$$

understanding acl in the PAC-field sense. Once again, the interpretability of the structure $F^{M_1(i)} \cup F^{M_0(i)}$ in $F^{M_1} \cup F^{M_0} \upharpoonright_{\mathcal{L}_-}$ proves the remark. \square

The following lemma is key for much of the rest of the proof. We prove that the algebraic closure of a subset of F is in the field-definable closure of the union of F and a full submodel.

Lemma 4.1.10. *Let M be a model of a (bounded PRC) theory T and M_0 a full submodel. Suppose $F^{M_0} \subseteq C \subseteq F^M$, $acl(C) \cap F^M = C$ and $a \in$*

$\text{acl}(C)$ for some a . Then there exists $e \in \text{dcl}^{\text{field}}(a, C) \cap M_0$ such that $a \in \text{dcl}^{\text{field}}(e, C)$.

Proof. As with the previous two remarks we can prove the lemma using the analogue result for PAC-fields (lemma 1.5 in [Hru91]) and interpretability of $C(i)$ and $M_0(i)$ by quantifier free formulas in $\mathcal{L}_-(C)$ and $\mathcal{L}_-(M_0)$ respectively. \square

Proposition 4.1.11. *Let T, M and M_0 be as in the lemma above. Let A be an algebraically closed subset of M that contains M_0 . Then $T \cup \text{diag}(A)$ is complete (in the added language).*

Proof. The proof is the same as the proof of proposition 1.6 in [Hru91] once we change stationary formulas to (our corresponding) open subsets (in the order topology) of irreducible affine varieties and dcl to $\text{dcl}^{\text{field}}$. \square

The following corollaries follow straight from the proofs of corollaries 1.7, 1.8 and 1.9 in [Hru91].

Corollary 4.1.12. *T is model complete.*

Corollary 4.1.13. *A submodel M_0 of a model M of T is a full submodel if and only if it is algebraically closed, full and F^{M_0} is a PRC subset of M_0 .*

Corollary 4.1.14. *Let M be a model of T , M_0 a full submodel. Then algebraic closure (in \mathcal{L}_+) over M_0 in M and coincides with field algebraic closure over M_0 .*

We can now prove the theorem.

Proof. (of theorem 4.1.2)

Let M be a bounded large saturated PRC field, let $e \in \text{dcl}^{\text{eq}}(F^M)$ be an imaginary element and let N be an elementary submodel of F . By lemma 4.1.3 we know that $N(i)$ is an elementary submodel of $F(i)$. By [Hru91] corollary 3.2 (elimination of imaginaries for bounded PAC structures) e is interdefinable with some tuple $c \in F(i)$ in the structure $(F(i), a)_{a \in N}$. By interpretability of $F(i)$ in $F \upharpoonright \mathcal{L}_-$ we can find such a c in F which proves e.i. for $(F, a)_{a \in N}$.

Now, let \overline{N} be the real closure of N . By corollary 4.1.13 (N, \overline{N}) is a full submodel of M and by corollary 4.1.14 algebraic closure in M and in $M|\mathcal{L}$ coincide once we add constants for all the elements in N . Therefore, algebraic closure in $(F, a)_{a \in N}$ is the field algebraic closure so it satisfies Steinitz exchange property.

By theorem 3.2.1, T^* is \mathfrak{p} -minimal. But being \mathfrak{p} -minimal is a property that is invariant under adding constants, so T is \mathfrak{p} -minimal. \square

4.2. Model Companions of Large differential Fields of Characteristic 0. In [Tre], Marcus Tressl introduced a first order theory in the language

of differential rings with k derivatives called UC^3 with the property that if T is the theory of a differential field (in the language of differential rings) such that the restriction T^{field} of T to the language of fields is model complete and has large models, then $T^* := T^{field} \cup UC$ is the model companion of T . We prove in this section that if T^{field} is rosy, T^* is rosy. As a corollary, we prove that if T^{field} is stable then T^* is stable.

In this section we work with a theory T in the language of differential rings containing the theory of differential fields such that T^{field} is a model complete theory with large models. We assume the reader has familiarity with the results and definitions in [Tre].

Definition 4.1. *Given a complete type $p(x)$, let $p^{field}(x, x_1, x_2, \dots, x_k x_{11}, \dots)$ be the restriction of $p(x)$ to the language of rings obtained by replacing all the derivatives of x by free variables and let $p^{qf}(x)$ be the set of all quantifier free formulas in $p(x)$.*

Lemma 4.2.1. *Let $p(x)$ be a complete type over some set A . Then, if $p^{field}(\bar{x})$ is consistent with T^{field} and $p^{qf}(x)$ is a consistent differential type, there is some model $N \models T^*$ containing A such that $p(x)$ is realized in N .*

Proof. We can assume without loss of generality that A is a differential field.

By [Tre] theorem 7.1(ii), $p(x)$ is implied by $p^{field}(\bar{x}) \cup p^{qf}(x)$. Let A^c be the differential closure of A (see [McG00]) and let M^{field} be a model of T^{field} such that there is some tuple $\langle a, a_1, a_2, \dots, a_k, a_{11} \dots \rangle$ in M^{field} realizing $p^{field}(\bar{x})$. Let F be the subfield of M generated by $A \cup \{\bar{a}_I\}$.

Since A^c is differentially closed there is some $a' \in M^c$ realizing $p^{qf}(x)$. Let D be the differential subfield of M^c generated by $\{Aa'\}$. By definition a' satisfies $p^{qf}(x)$ so the tuple $\langle a', d_1 a', d_2 a', \dots, d_k a', d_1 d_1 a' \dots \rangle$ satisfies all the quantifier free formulas that appear in p^{field} and we have a map of rings sending $\langle A, a', d_1 a', d_2 a', \dots, d_k a', d_1 d_1 a' \dots \rangle$ to $\langle A, a, a_1, a_2, \dots, a_k, a_{11} \dots \rangle$. We can use this map to equip F with k commuting derivatives extending those in A in such way that $a \models p^{qf}(x)$. By [Kol73] we can define k commuting derivatives in M^{field} extending those in F so there is some differential field M such that F is a subfield of M and such that M^{field} is the restriction of M to the language of rings. By [Tre] theorem 6.2(II) there is a differential field L extending M satisfying T^* . By model completeness of T^{field} , the field type of a over A is the same in M as it is in L so $L \models p^{field}(a)$.

Therefore, $L \models p^{field}(a) \cup p^{qf}(a)$ and $L \models p(a)$. \square

Corollary 4.2.2. *Let T, T^*, T^{field} be as above, M be a model of T^* , $A \subset B$ subsets of M . Let $p(x, B)$ be a type over B realized by some a and let k -DCF be the theory of closed differential fields with k commuting derivatives. Then,*

³ UC is a system of axioms that basically says that any algebraically prepared system (a system of differential polynomials that is consistent with T^{field}) has a realization.

if T^* implies that $p(x)$ strongly divides over A , either T^{field} implies p^{field} strongly divides over A or k -DCF implies p^{qf} strongly divides over A ⁴.

Proof. Suppose $p(x)$ strongly divides over A in the sense of T^* and let $q(Y, A)$ be the type of B over A . By definition, there is some $n \in \mathbb{N}$ such that

$$\bigcup_{i=1}^n p(x, Y_i) \cup \bigcup_{i=1}^n q(Y_i, A)$$

is inconsistent with T^* and $q(Y, A)$ is non algebraic. By lemma 4.2.1, either

$$\bigcup_{i=1}^n p^{field}(x, Y_i) \cup \bigcup_{i=1}^n q^{field}(Y_i, A)$$

is inconsistent with T^{field} , or

$$\bigcup_{i=1}^n p^{qf}(x, Y_i) \cup \bigcup_{i=1}^n q^{qf}(Y_i, A)$$

is inconsistent with k -DCF. Since on one hand $tp^{field}(B/A)$ is precisely $q^{field}(Y, A)$ and on the other $q^{qf}(Y, A)$ is quantifier free and we have elimination of quantifiers in k -DCF, the result follows. \square

Remark 4.2.3. Note that strong dividing implies forking so, by [McG00], if k -DCF implies p^{qf} strongly divides over A then the Δ -differential rank of the prime ideal \mathcal{I}_p generated by differential polynomials in p is smaller than the differential rank of the prime ideal $\mathcal{I}_{p|A}$ generated by the differential polynomials in p with coefficients in A .

Theorem 4.2.4. If T^{field} is rosy so is T^* .

Proof. Let \mathcal{M} be a monster model of T^* , let $A \subset B$ be (small) subsets of \mathcal{M} and let $p(x)$ be a type over B which \mathfrak{p} -forks over A . By definition of \mathfrak{p} -forking this is witnessed by

$$p(x) \vdash \bigvee_{i=1}^n \phi_i(x, b_i)$$

where for each of i we have that $\phi_i(x, b_i)$ strongly divides over Ac_i for some c_i ; let $D := \bar{b}_i$.

Claim 4.2.5. Let $p^{field}(x)$ and $p^{qf}(x)$ be as above. Then either $p^{field}(x)$ \mathfrak{p} -forks over A in the sense of T^{field} or $p^{qf}(x)$ \mathfrak{p} -forks over A in the sense of k -DCF.

⁴If we look in the differential closure of M the quantifier free type of a over B is precisely p^{qf} ; by p^{qf} strongly dividing over A in the sense of k -DCF we mean that in the differential closure of M the quantifier free type of a over B strongly divides over A . This is equivalent to say that the differential ideal of p^{qf} has smaller differential rank than the differential ideal of $p^{qf}|_A$ (see remark below).

Proof. Let $p^{field}(x, D)$ and $p^{qf}(x, D)$ be types over $B \cup D$ which are, respectively, non \mathfrak{p} -forking extensions of $p^{field}(x)$ and $p^{qf}(x)$ in the sense of T^{field} and k -DCF. By lemma 4.2.1 the type $p^{field}(x, D) \cup p^{qf}(x, D)$ is consistent with T^* so it is realized by some $a \in M$. By [Tre] theorem 7.1(ii) we know that $p^{field}(x, D) \cup p^{qf}(x, D)$ implies $tp(a/BD)$ so by construction there is some i such that a realizes $\phi_i(x, b_i)$. This implies that $tp(a/BD)$ strongly divides over Ac_i . By theorem 4.2.2, either $p^{field}(x, D)$ \mathfrak{p} -divides over A in the sense of T^{field} or $p^{qf}(x, D)$ \mathfrak{p} -divides over A in the sense of k -DCF. By construction neither of them \mathfrak{p} -forks over B so transitivity of \mathfrak{p} -forking implies that either $p^{field}(x)$ \mathfrak{p} -forks over A in the sense if T^{field} or $p^{qf}(x)$ \mathfrak{p} -forks over A in the sense of differentially closed fields. \square

To finish the prove of the theorem, just note that if we had an infinite \mathfrak{p} -forking chain in a model of T^* we would have an infinite \mathfrak{p} -forking chain in a model of T^{field} or some differential ideal with infinite differential rank. Since T^{field} is rosy and k -DCF is stable, this cannot happen. \square

Corollary 4.2.6. *If T^{field} is stable so is T^* .*

Proof. Let M be a monster model of T^* . Since \mathfrak{p} -forking is an independence relation it is enough to show that given a small model $N \models T^*$ a type $p(x)$ over N and some tuple a there is a unique non \mathfrak{p} -forking extension of $p(x)$ to Na . Let $q(x, a)$ and $r(x, a)$ be two non \mathfrak{p} -forking extensions of $p(x)$. Let $q^{field}, r^{field}, p^{field}, q^{qf}, r^{qf}$ and p^{qf} be the types obtained by restricting types q, r and p to the language of rings and to the quantifier free formulas. By quantifier elimination and stability of k -DCF we have that $r^{qf} = q^{qf}$ and by stability of T^{field} (and therefore uniqueness of non \mathfrak{p} -forking extensions) $r^{field} = q^{field}$. Since both $q(x, a)$ and $r(x, a)$ are implied by $q^{field} \cup q^{qf}$ and $r^{field} \cup r^{qf}$ respectively, we conclude that $r(x, a) = q(x, a)$ so that there is a unique non \mathfrak{p} -forking extension of $p(x)$ to Na . \square

REFERENCES

- [Cha99] Zoé Chatzidakis. Simplicity and independence for pseudo-algebraically closed fields. In *Models and computability (Leeds, 1997)*, volume 259 of *London Math. Soc. Lecture Note Ser.*, pages 41–61. Cambridge Univ. Press, Cambridge, 1999.
- [CP98] Z. Chatzidakis and A. Pillay. Generic structures and simple theories. *Ann. Pure Appl. Logic*, 95(1-3):71–92, 1998.
- [GvdDM81] Gregory Gherlin, Lou van den Dries, and Angus Macintyre. Decidability and undecidability theorems for pac-fields. *Bull. Amer. Math. Soc.*, 4(1):101–104, 1981.
- [HKP00] Bradd Hart, Byunghan Kim, and Anand Pillay. Coordinatisation and canonical bases in simple theories. *J. Symbolic Logic*, 65(1):293–309, 2000.
- [Hru91] Ehud Hrushvski. Pseudo-finite fields and related structures. *manuscript*, 1991.
- [Kim96] Byunghan Kim. *Simple First Order Theories*. PhD thesis, University of Notre Dame, 1996.
- [Kol73] E.R. Kolchin. Differential algebra and algebraic groups. *Pure and Applied Mathematics*, 54, 1973.

- [KP01] Byunghan Kim and A. Pillay. Around stable forking. *Fund. Math.*, 170(1-2):107–118, 2001. Dedicated to the memory of Jerzy Łoś.
- [McG00] Tracey McGrail. The model theory of differential fields with finitely many commuting derivatives. *J. Symbolic Logic*, 65(2):885–913, 2000.
- [Ons] Alf Onshuus. Properties and consequences of th-forking. *Preprint*.
- [Ons02] Alf Onshuus. *Thorn-Forking in Rosy Theories*. PhD thesis, University of California at Berkeley, 2002.
- [Pre81] Alexander Prestel. Pseudo real closed fields. In *Set theory and model theory*, volume 872 of *Lecture Notes in Math.*, pages 127–156. Springer, Berlin, 1981.
- [Sch00] Joachim Schmid. A density property for pseudo real closed fields. *Konstanzer Schriften in Mathematik und Informatik*, (124), 2000.
- [Tre] Marcus Tressl. A uniform companion for large differential fields of characteristic 0. *Preprint*.
- [vdD98] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.
- [Wag00] Frank O. Wagner. *Simple theories*, volume 503 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000.